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Amplitude ratios and β estimates from general dimension percolation moments

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Abstract. Low concentration series are generated for moments of the percolation cluster size distribution, $\Gamma_j = \langle s^{j-1} \rangle$ (s is the number of sites on a cluster) for $j = 2, \ldots, 8$ and general dimensionality d. These diverge at p_c as $\Gamma_j \sim A_j(p_c - p)^{-\gamma_j}$ with $\gamma_j = \gamma_j = \gamma + (j-2)\Delta$, where $\Delta = \gamma + \beta$ is the gap exponent. The series yield new accurate values for Δ and β, $\Delta = 2.23 \pm 0.05, 2.10 \pm 0.04, 2.03 \pm 0.05$ and $\beta = 0.44 \pm 0.15, 0.66 \pm 0.09, 0.83 \pm 0.08$ at $d = 3$, 4, 5. In addition, ratios of the form A_jA_k/A_mA_n, with $j + k = m + n$, are shown to be universal. New values for some of these ratios are evaluated from the series, from the ε expansion ($\varepsilon = 6 - d$) and exactly (in $d = 1$ and on the Bethe lattice). The results are in excellent agreement with each other for all dimensions. Results for different lattices at $d = 2$, 3 agree very well. These amplitude ratios are much better behaved than other ratios considered in the past, and should thus be more useful in characterising percolating systems.

1. Introduction

Despite the extensive literature on exact power series expansions for percolation (see Essam (1972) for an introduction to the series derivations and Adler *et al* (1983) for more recent analyses) there remain several aspects of percolation concerning which few or no series results have been obtained to date. In particular, there are no direct estimates of the exponents β of the percolation probability and Δ ($= \gamma + \beta$), the so-called 'gap exponent', for dimensions $d \geq 4$. Furthermore, in all dimensions there are very few series estimates of *critical amplitude ratios* and most of the existing ones have very large uncertainties.

As regards the exponent β, the values which exist in the literature are rather unsatisfactory. As seen in table 1, values quoted in the review by Stauffer (1979) strongly disagree with those obtained from the ε expansion and with indirect estimates based on series and scaling relations (from de Alcantara Bonfim *et al* (1980, 1981), Adler (1984) and Adler *et al* (1985)).

Universal amplitude ratios were reviewed in detail by Aharony (1980) who also obtained ε expansions for them. Some agreement was found between some series estimates at $d = 2$ and extrapolations of the ε expansion, but results for some other amplitude ratios (e.g. C^+/C^- for the mean cluster size below and above p_c) varied considerably, particularly in Monte Carlo simulations and were difficult to extrapolate from the ε expansion down to $d = 2$, 3. It is thus desirable to find universal amplitude ratios which are less sensitive. In principle, percolation systems may belong to different universality classes, e.g., depending on the range of correlations among the occupation probabilities. Amplitude ratios should play an equal role to that played by critical

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Table 1. Estimates for the exponents Δ and β.

d	5	4	3
Δ (our series)	2.03 ± 0.05	2.10 ± 0.04	2.23 ± 0.05
Δ (ε expansion)[b]	2.02 ± 0.005	2.08 ± 0.02	2.16 ± 0.04
γ (previous series)	1.02 ± 0.03[a]	1.44 ± 0.05[a]	1.79 ± 0.10
$\beta = \Delta - \gamma$	0.83 ± 0.08	0.66 ± 0.09	0.44 ± 0.15
β from series Γ_3/Γ_2^2	0.83 ± 0.1	0.67 ± 0.1	0.44 ± 0.1
β (ε expansion)	0.835 ± 0.005	0.64 ± 0.02	0.34 ± 0.04
β (mainly MC[c])	0.7	0.5	
β (from RFIM)[d]	0.84	0.64	
β (Jan *et al*)[e]	0.67	0.56	
β (Grassberger)[f]		0.65 ± 0.04	0.43 ± 0.04

[a] From Adler *et al* (1984).
[b] ε expansion, calculated from results of de Alcantara Bonfim *et al* (1981) using their ν and γ estimates and scaling.
[c] From Stauffer (1979).
[d] Deduced via scaling from the results of Adler *et al* (1985) for the random field Ising model and the dilute antiferromagnet.
[e] Jan *et al* (1985).
[f] Grassberger (1986).

exponents in identifying the universality class of a given system. In particular, both should be studied in realistic continuum porous media in order to find out if these belong to the same universality class as the uncorrelated bond or site percolation for which most theoretical calculations have been done.

In the present paper we pursue these aims by undertaking a comprehensive study of the moments Γ_j of the percolation cluster size distribution. If $n_s(p)$ is the probability of a site (at concentration p of sites or of bonds) belonging to a cluster of s sites, then

$$\Gamma_j = \langle s^{j-1} \rangle = \sum_s s^j n_s(p). \tag{1.1}$$

Using the 'ghost' field H, Γ_j can be derived as

$$\Gamma_j = \left(\frac{\partial}{\partial H}\right)^j \sum_s n_s(p)\, \mathrm{e}^{-sH}\big|_{H=0} = -\left(\frac{\partial}{\partial H}\right)^{j-1} [1 - P_\infty(p, H)]\big|_{H=0} \tag{1.2}$$

where $P_\infty(p, H)$ is the probability of a site belonging to the infinite cluster. In § 2 we use scaling arguments to show that, for $p < p_c$,

$$\Gamma_j \simeq A_j (p_c - p)^{-\gamma_j} [1 + a_j (p_c - p)^{\Delta_1} + \ldots] \tag{1.3}$$

with

$$\gamma_j = \gamma + (j-2)\Delta \tag{1.4}$$

where γ describes the divergence of the mean cluster size Γ_2, while $\Delta = \gamma + \beta$, with $P_\infty \propto (p - p_c)^\beta$ at $H = 0$, $p > p_c$. The validity of equation (1.4) was implied by the renormalisation group of Harris *et al* (1975) and was proven for the Bethe lattice by Essam *et al* (1976), who also present numerical evidence to support it in $d = 2$ and $d = 3$. The exponent Δ_1, expected to be the same for all j, represents the leading confluent correction.

Section 2 also shows that amplitude ratios of the form $A_j A_k / A_m A_n$, with $j + k = m + n$, should be universal. The ε expansion is then used, in § 3, to estimate $A_2 A_4 / A_3^2$,

A_3A_5/A_4^2, $A_2^2A_5/A_3^3$ and A_2A_5/A_3A_4 and the results are summarised in table 2 and figure 1. Exact calculations, both at $d=1$ and on the Bethe lattice, are described in §4.

Section 5 describes our derivation of the low concentration series for Γ_j and their analysis which yields the gap exponent Δ. The analysis of series for quantities such as Γ_3/Γ_2^2 yields *direct* estimates for the exponent β. Alternatively, β can be obtained from $\beta=\Delta-\gamma$ using values obtained for Δ and γ from our series. Our results are summarised and compared with alternative evaluations in table 1. The agreement with the ε expansion values is excellent.

It turns out that series estimates of the individual amplitudes A_j are not very accurate. However, the universal combination A_jA_k/A_mA_n can be obtained directly from series for $\Gamma_j\Gamma_k/\Gamma_m\Gamma_n$, which should have a regular leading behaviour near p_c. Our series analysis of these ratios is described in § 6 and the results are shown in figure 1. We observe that the agreement between the series and the ε expansion values is extremely good for all ratios and all dimensions. This agreement is significant in view of doubts one might have that a ϕ^3 field theory (Fucito and Marinari 1981, Fucito and Parisi 1981) might not be suitable for an ε expansion.

After completing these calculations we received a preprint from Grassberger (1985) and became aware of a letter of Jan *et al* (1985) who calculate β in four and four, five dimensions respectively. Grassberger (1985) finds $\beta=0.62$ in four dimensions (corresponding to $p_c=0.1583\pm0.0002$) but does not make any allowance for corrections to scaling. We propose to investigate this discrepancy in the future†. Jan *et al* (1985) find $\beta=0.67$ ($d=5$) and $\beta=0.56$ ($d=4$), somewhat below our and the ε expansion estimates and close to the old Monte Carlo values. They also find lower ν values than the ε expansion (de Alcantara Bonfim *et al* 1981), $\nu=0.51$ ($d=5$, cf 0.57) and $\nu=0.64$ ($d=4$, cf 0.68). Thus their final $d_f=d-\beta/\nu$ values, 3.69 ($d=5$) and 3.12 ($d=4$), are not all that different from values calculated from our β and ε expansion ν values, viz $d_f=3.53$ ($d=5$) and $d_f=3.06$ ($d=4$).

2. Scaling

As explained in detail by Aharony (1980), $P_\infty(p,H)$ must obey the asymptotic scaling form

$$H/P_\infty^\delta=h(t/P_\infty^{1/\beta}) \tag{2.1}$$

where $t=(p_c-p)/p_c$, P_∞ and H are all small. The function $y=h(X)$ contains two non-universal parameters, h_0 and X_0, defined via $h_0=h(0)$ and $h(-X_0)=0$. Rescaling h by h_0 and X by X_0, the resulting equation of state

$$\tilde{h}(\tilde{X})=\tilde{h}(X/X_0)=h_0^{-1}h(X) \tag{2.2}$$

is *universal*. All the critical amplitudes may be related to X_0 and h_0 and combinations of them in which X_0 and h_0 cancel are thus also universal.

Solving equation (2.2) for P_∞ one finds

$$P_\infty(t,H)=(t/X_0)^\beta\tilde{f}(X_0^\Delta H/h_0t^\Delta) \tag{2.3}$$

where $\Delta=\delta\beta=\beta+\gamma$ and where $\tilde{f}$ is a *universal* function. The individual details of a specific problem, e.g. site or bond percolation or (short) range of correlations, enter only into X_0 and h_0. Taking derivatives of (2.3), we now find that the leading divergence of Γ_j is indeed described by the exponents γ_j of equation (1.4). Also, we identify

$$A_j=(-X_0^\Delta/h_0)^{j-1}X_0^{-\beta}\tilde{f}^{(k-1)}(0). \tag{2.4}$$

† See note added in proof.

Clearly this implies that when $k+j=m+n$ one has

$$A_jA_k/A_mA_n=\tilde{f}^{(j-1)}\tilde{f}^{(k-1)}/\tilde{f}^{(m-1)}\tilde{f}^{(n-1)} \tag{2.5}$$

and the RHS is universal. In what follows we shall thus ignore the factors of X_0 and h_0 and concentrate on the universal functions $\tilde{h}$ and $\tilde{f}$.

For $t>0$ ($p<p_c$) we have $P_\infty\to 0$ as $H\to 0$. Thus the argument $X=t/P_\infty^{1/\beta}$ in (2.1) is infinite. As discussed by Aharony (1980), the function $h(X)$ has the large-X expansion

$$h(X)=\sum_{n=1}^{\infty}\eta_n X^{\gamma-(n-1)\beta}. \tag{2.6}$$

From equation (2.1)

$$\partial H/\partial P_\infty=t^\gamma X^{-\gamma}[\delta h-(1/\beta)Xh'(X)]. \tag{2.7}$$

Thus

$$\Gamma_2=\partial P_\infty/\partial H=t^{-\gamma}X^\gamma/[\delta h-(1/\beta)Xh'(X)]. \tag{2.8}$$

Using (2.5), this becomes

$$\Gamma_2=t^{-\gamma}\left(\sum_n n\eta_n X^{-(n-1)\beta}\right)^{-1} \tag{2.9}$$

and we identify $A_2=1/\eta_1$. It is now straightforward to take further derivatives and to find

$$\begin{aligned}
A_3&=2\eta_2/\eta_1^3\\
A_4&=(12\eta_2^2-6\eta_1\eta_3)/\eta_1^5\\
A_5&=(24\eta_4\eta_1^2-120\eta_3\eta_2\eta_1+120\eta_2^3)/\eta_1^7\\
A_6&=(720\eta_4\eta_2\eta_1^2-120\eta_5\eta_1^3+360\eta_3^2\eta_1^2-2520\eta_3\eta_2^2\eta_1+1680\eta_2^4)/\eta_1^9
\end{aligned} \tag{2.10}$$

etc.

So far we discussed only the asymptotic form. Denoting the leading irrelevant variable by w, equation (2.3) should be replaced by

$$P_\infty(t,H,w)=(t/X_0)^\beta\tilde{f}[X_0^\Delta H/h_0t^\Delta,\ wt^{\Delta_1}] \tag{2.11}$$

where Δ_1 is the exponent associated with the renormalisation group flow of w towards its fixed point value of zero. The function $\tilde{f}$ is still universal and the only additional non-universal amplitude concerns the magnitude of w.

Taking derivatives of (2.11) with respect to H will now yield

$$\begin{aligned}
\Gamma_j&=A_jt^{-\gamma_j}\tilde{f}^{(j-1)}(0,wt^{\Delta_1})\\
&\simeq A_jt^{-\gamma_j}[\tilde{f}^{(j-1)}(0,0)+\tilde{f}^{(j-1,1)}(0,0)wt^{\Delta_1}+\ldots]
\end{aligned} \tag{2.12}$$

where the coefficient $\tilde{f}^{(j-1,1)}(0,0)$ is again universal. This explains the general form (1.3) and implies that ratios like a_j/a_k are also universal (Aharaony 1980).

3. Epsilon expansion

Aharony (1980) used the $q\to 1$ limit of the q-state Potts model, equivalent to the bond percolation problem, to derive ε expansions in $d=6-\varepsilon$ dimensions of the function

$\tilde{h}(\tilde{X})$ and for the coefficients η_1, η_2 and η_3. Extending those results to the next term (and setting $X_0 = h_0 = 1$), they may be summarised as

$$\begin{aligned}
\eta_1 &= 2^{2-\delta} + O(\varepsilon^3) \\
\eta_2 &= \eta_1\left(1 + \tfrac{2}{7}\varepsilon + \frac{565}{2\times 3^2 7^3}\varepsilon^2\right) + O(\varepsilon^3) \\
\eta_3 &= \eta_1 \tfrac{2}{7}\varepsilon\left(1 + \frac{817}{2^2 3^2 7^2}\varepsilon\right) + O(\varepsilon^3) \\
\eta_4 &= -\eta_1 \tfrac{4}{21}\varepsilon\left(1 - \frac{191}{2^2 3^2 7^2}\varepsilon\right) + O(\varepsilon^3)
\end{aligned} \tag{3.1}$$

where

$$\delta = 2 + \tfrac{2}{7}\varepsilon + \frac{565}{2\times 3^2 7^3}\varepsilon^2 + O(\varepsilon^3).$$

The results for $d > 6$ stick to the mean-field (or Bethe lattice) values, obtained by setting $\varepsilon = 0$ in these expressions.

Substituting in equations (2.10), this yields (up to order ε^2)

$$\begin{aligned}
\frac{A_2 A_4}{A_3^2} &= 3\left(1 - \tfrac{1}{7}\varepsilon + \frac{191}{2^2 3^2 7^3}\varepsilon^2\right) \\
\frac{A_2^2 A_5}{A_3^3} &= 15\left(1 - \tfrac{34}{105}\varepsilon + \frac{12542}{5\times 2^2 3^3 7^3}\varepsilon^2\right) \\
\frac{A_2 A_5}{A_3 A_4} &= 5\left(1 - \tfrac{19}{105}\varepsilon + \frac{4889}{5\times 2^2 3^3 7^3}\varepsilon^2\right) \\
\frac{A_3 A_5}{A_4^2} &= \tfrac{5}{3}\left(1 - \tfrac{4}{105}\varepsilon + \frac{1016}{5\times 2^2 3^3 7^3}\varepsilon^2\right).
\end{aligned} \tag{3.2}$$

Table 2 contains estimates of these ratios for various dimensions, based on different Padé estimates. We note that the ε expansion of the ratio A_3^2/A_2A_4 is the same (to order ε^2) as that of $(2-\beta)/3$. In the fourth row of table 2 we thus list values of $(2-\beta)/3$, using estimates of β from table 1.

Apart from $A_2^2A_5/A_3^3$, all the Padé estimates agree reasonably well with each other and we used a (subjective) average to represesnt them in figure 1. The coefficients in the ε expansion of $A_2^2A_5/A_3^3$ are rather large, and therefore some of the Padé estimates are not reasonable. We list values only for the estimate which looks similar to the series. Note that the amplitude ratios listed here are not all independent of each other. For example,

$$\frac{A_2^2 A_5}{A_3^3} = \left(\frac{A_2 A_4}{A_3^2}\right)\left(\frac{A_2 A_5}{A_3 A_4}\right) = \left(\frac{A_3 A_5}{A_4^2}\right)\left(\frac{A_2 A_4}{A_3^2}\right)^2. \tag{3.3}$$

Thus, one may choose the better behaved ε expansions (e.g. for A_2A_4/A_4^2) and derive the others from them.

We now turn to the correction terms, equation (1.3) or (2.12). As discussed in detail by Aharony (1980) and Aharony and Ahlers (1980), the renormalisation group equations to leading order in ε always yield results of the form

$$\Gamma_j = A_j t^{-\gamma_j}\left(1 + w t^{\Delta_1}\right)^{2(\gamma_j - \gamma_j^0)/\varepsilon} \tag{3.4}$$

Table 2. Estimates of amplitude ratios.

Ratio	Estimate	Dimension 5	4	3	2	1
A_2A_4/A_3^2	$3(1-0.1429\varepsilon+0.015\,47\varepsilon^2)$	2.62	2.33	2.13	2.03	2.02
	$3/(1+0.1429\varepsilon+0.005\varepsilon^2)$	2.61	2.30	2.04	1.82	1.63
	$3(1-0.0346\varepsilon)/(1+0.1083\varepsilon)$	2.61	2.30	2.03	1.80	1.61
	$3/(2-\beta)$	2.56	2.24	1.92	1.61	1.5
	Series (hypercubic)	2.62	2.30	1.94	1.61	$\frac{4}{3}$[a]
	Largest approximant	2.70	2.73	2.04	1.69	1.333
	Smallest approximant	2.58	2.15	1.88	1.59	1.333
	Series (FCC, triangular)	—	—	2.0	1.72	—
A_3A_5/A_4^2	$\frac{5}{3}(1-0.0381\varepsilon+0.005\,49\varepsilon^2)$	1.61	1.58	1.56	1.56	1.58
	$\frac{5}{3}(1+0.0381\varepsilon-0.004\,03\varepsilon^2)$	1.61	1.57	1.55	1.53	1.53
	$5(1+0.106\varepsilon)/3(1+0.144\varepsilon)$	1.61	1.57	1.53	1.51	1.48
	Series (hypercubic)	1.60	1.55	1.46	1.36	$\frac{5}{4}$[a]
	Largest approximant	1.60	1.55	1.55	1.47	1.250
	Smallest approximant	1.60	1.55	1.42	1.10	1.250
A_2A_5/A_4A_3	$5(1-0.181\varepsilon+0.0264\varepsilon^2)$	4.23	3.71	3.46	3.49	3.78
	$5/(1+0.181\varepsilon+0.0067\varepsilon^2)$	4.21	3.60	3.12	2.73	2.41
	$5(1-0.0373\varepsilon)/(1+0.1437\varepsilon)$	4.21	3.59	3.10	2.70	2.37
	Series (hypercubic)	4.15	3.4	2.84	2.12	$\frac{5}{3}$[a]
	Largest approximant	4.20	3.63	3.08	2.36	1.666
	Smallest approximant	4.10	2.78	2.71	1.48	1.666
$A_2^2A_5/A_3^3$	$15/(1+0.3238\varepsilon+0.0371\varepsilon^2)$	11.02	8.35	6.5	5.19	4.23
	Series (hypercubic)	10.80	8.12	5.48	3.65	$\frac{20}{9}$[a]
	Largest approximant	11.20	10.20	5.81	3.98	2.222
	Smallest approximant	10.50	7.30	4.21	2.10	2.222

[a] Exact.

where γ_j^0 is the mean-field value of γ_j. For small t, the RHS can be expanded and we identify a_j in equation (1.3) as

$$a_j = 2w(\gamma_j - \gamma_j^0)/\varepsilon + O(\varepsilon) \tag{3.5}$$

i.e.

$$a_j/a_k = (\gamma_j - \gamma_j^0)/(\gamma_k - \gamma_k^0). \tag{3.6}$$

To order ε, $\gamma_j = \gamma + (j-2)\Delta = 2j - 3 + \varepsilon/7$ and thus $\gamma_j - \gamma_j^0 = \varepsilon/7$, independent of j. We therefore conclude that

$$a_j/a_k = 1 + O(\varepsilon). \tag{3.7}$$

In particular, combinations like $\Gamma_j\Gamma_k/\Gamma_m\Gamma_n$, with $j+k=m+n$, will have no correction to scaling to this leading order (the coefficient of t^{Δ_1} would involve $a_j + a_k - a_m - a_n = O(\varepsilon)$).

4. Exact results

In one dimension and on a Bethe lattice, which should correspond to dimensions larger than six, the upper critical dimension, we were able to find the amplitude ratios

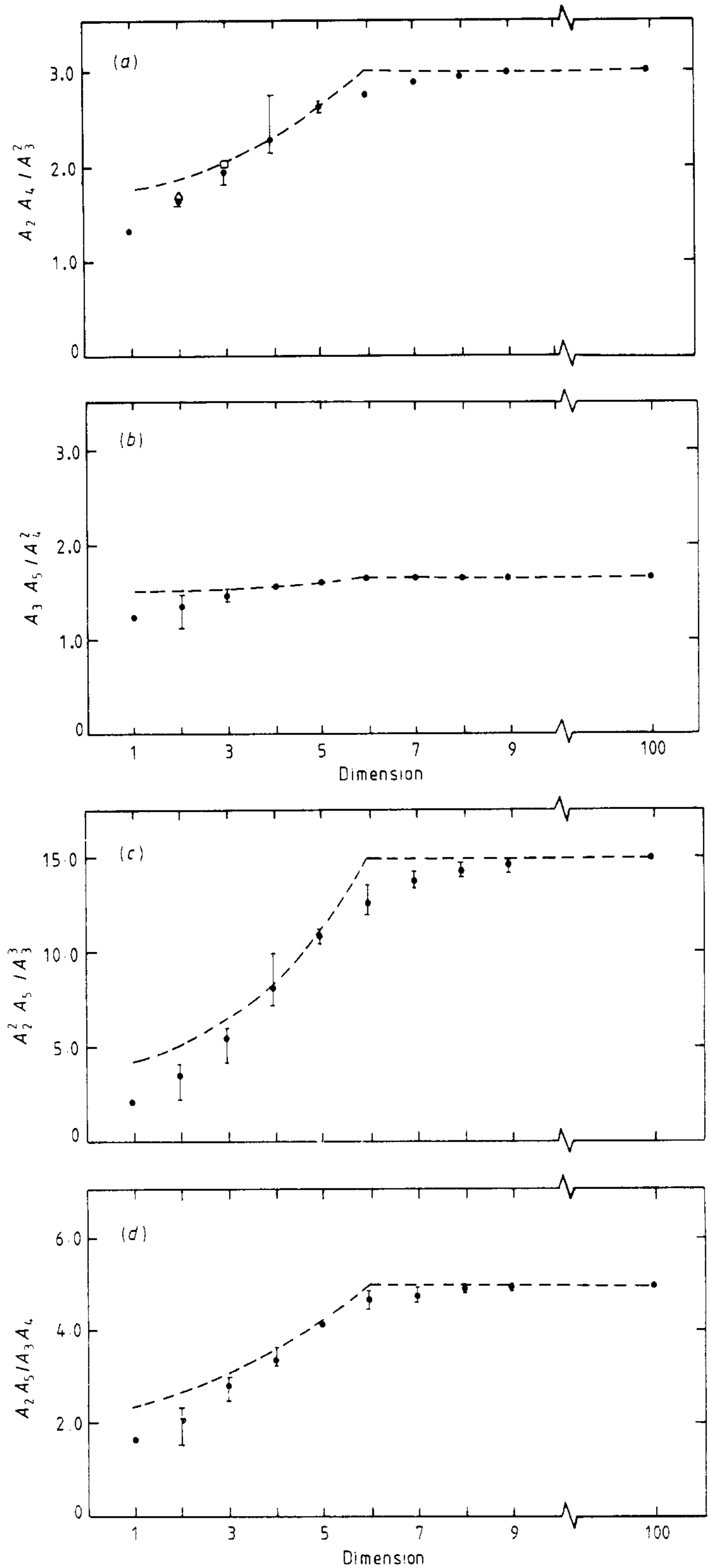

Figure 1. Amplitude ratios. Broken curve based on ε expansion; ●, hypercubic lattices; △, triangular lattice and □, FCC lattice. (a), A_2A_4/A_3^2; (b), A_3A_5/A_4^2; (c), $A_2^2A_5/A_3^3$; (d), A_2A_5/A_3A_4.

exactly. In one dimension, the 'free energy' is easily calculated:

$$f(p, H)=\sum_s p^2(1-p)^2 \mathrm{e}^{-sH}=\frac{(1-p)^2}{1-p\,\mathrm{e}^{-H}} \tag{4.1}$$

and one can find the amplitudes explicitly:

$$A_n=n!. \tag{4.2}$$

On a Bethe lattice of coordination σ the free energy is (Fisher and Essam 1961)

$$f(p, H)=\sum_s (\sigma+1)(1-p)^2\frac{(s\sigma)!}{s!(s\sigma-s+2)!}[p(1-p)^{\sigma-1}]^s \mathrm{e}^{-sH}. \tag{4.3}$$

A tedious but straightforward calculation leads to

$$\begin{aligned} A_2&=\tfrac{1}{2}\\ A_3&=\tfrac{1}{2}\times(-\tfrac{1}{2})\\ A_4&=\tfrac{1}{2}\times(-\tfrac{1}{2})\times(-\tfrac{3}{2}) \end{aligned} \tag{4.4}$$

etc, up to a multiplicative constant which depends on σ. We can see that the amplitude ratios correspond to those obtained by Aharony (1980) in six dimensions and to the $\varepsilon=0$ values in equation (3.2).

5. Series exponents

In this section we analyse low concentration series for $\Gamma_j=\langle s^{j-1}\rangle$, where s is the number of sites in bond percolation clusters (SB). The derivation of the series, via lattice animal data, is a straightforward extension of the case $j=2$, described in detail by Fisch and Harris (1978) and analysed by Adler *et al* (1984)†.

The series take the general form

$$\Gamma_j=1+\sum_{i,k} G_j(i, k)d^k p^i \tag{5.1}$$

and the coefficients $G_j(i, k)$ are presented in the appendix, for $j=3$-8.

The individual series were analysed by the method of Adler *et al* (1983), using as input the values of p_c and Δ_1 from Adler *et al* (1984) for $d>4$ and $p_c=0.2486$ for $d=3$ (Grassberger, private communication). All the series are expected to diverge at the same p_c.

We first analysed series for individual series and table 3 lists our estimates for γ_j, $j=2,\ldots,5$. From these we deduced our direct estimates of Δ, listed in table 1. These were then combined with known values of γ to derive $\beta=\Delta-\gamma$. In addition, we derived and analysed the series for Γ_3/Γ_2^2, which should diverge as $(p_c-p)^{-\beta}$.

Our value of Δ at $d=3$ agrees well with that quoted by Essam and Gwilym (1971), $\Delta=2.2\pm0.3$. Our various estimates for β agree well with each other and with those from the ε expansion. All these estimates disagree with the previously quoted Monte Carlo values (Stauffer 1979). While new Monte Carlo estimates would be nice to confirm our resolution of this discrepancy it seems quite clear that estimates of $\beta\sim0.8$ and $\beta\sim0.65$ for $d=5$ and $d=4$ percolation should be quoted in future.

† Note that the series for the number of bonds in a bond cluster (BB), quoted by Adler *et al* (1984), should not include a constant (p-independent) term. The analysis of this series was, however, correct. We also note that the BB series presented there are an extension of the Gaunt and Ruskin (1978) series and the SB series an extension of the Fisch and Harris (1978) series.

Table 3. Estimates for γ_j (errors are of order ±0.10).

d	>6	5	4	3
γ_2	1	1.19	1.44	1.79
γ_3	3	3.24	3.51	4.05
γ_4	5	5.26	5.64	6.27
γ_5	7	7.29	7.77	8.49

6. Series amplitude ratios

We used the series for Γ_j to generate series for ratios like $\Gamma_j\Gamma_k/\Gamma_m\Gamma_n$, for the four cases listed in table 2 and shown in figure 1. Our new series were used for hypercubic lattices at $d = 1, 2, \ldots, 9$ and $d = 100$, and the series given by Essam *et al* (1976) were used for $\Gamma_2\Gamma_4/\Gamma_3^2$, counting bonds on bond clusters on the triangular and FCC lattices. We note that the FCC series are rather short (to order p^9 only) but nevertheless we attempted their analysis. We then used direct Padé approximants for the ratio series to estimate their values at $p = p_c$. We note that this method does not require any assumptions about exponent values. Also, p_c is used only once in the calculations and the results are not very sensitive to small variations in its value. This probably results from the fact that the leading singularity was divided out and that correction terms to the ratio are very small (as predicted in § 3). Indeed, we were not successful in our attempts to identify such corrections by direct D log Padé analysis of derivatives of the ratio series. For comparison purposes we also tried to evaluate the A_j for each moment series individually (using the method described by Gaunt and Guttman (1974)), but found that owing to the large uncertainties in γ and β the errors were large and the convergence was rather poor.

The results are plotted in figure 1 and some are presented in detail in table 2. We have evaluated nine central and near-diagonal approximants to each ratio, discarding those with obvious defects and averaging over the remaining ones. (We never needed to discard more than two approximants and in most cases none were discarded.) We quote the central values and most extreme approximants in the table for each ratio and dimension on the hypercubic lattices. In the figure we indicate central values by a filled circle and extreme approximants by error bars. Where no error bars are present on the graph it is because the convergence errors are smaller than the size of the dot. We note that the estimate on the triangular lattice has error bars comparable to those on the square lattice†. The approximants to the FCC lattice ratio, however, have quite a wide range and the estimate ~2.0 comes from a choice of five approximants (2.09, 2.14, 2.014, 1.989 and 1.886), whereas an additional four (−0.875, 2.77, 3.9 and 2.81) give an average of 2.2. The poor convergence is presumably due to the shortness of the series.

Explicit analysis of our amplitude ratio series at $d = 1$ and $d = 100$ gave excellent agreement with the exact results of § 4 and served as a confirmation of the reliability of our series evaluation. As can be seen from table 2 and figure 1, the results in other dimensions also show excellent agreement between different lattices and with the ε expansion. However, we note that the finite series cannot reproduce the sharp break at $d = 6$. Instead, the series values already begin to deviate slightly from the mean-field

† The central estimate is 1.72 with a range 1.67–1.75.

values at $d = 9$. In spite of all this, even the maximum deviation, at $d = 6$, is rather small. Similar roundings were observed for critical exponents and are partly caused by the failure of the (relatively) short series to respond to the logarithmic corrections at $d = 6$. As indicated by the error bars on the graph, the different Padé approximants are quite close here, this being a systematic error. For $d = 3$ and $d = 4$ the ε expansion results fall well within the range of Padé approximants. As might be expected, the agreement between the series and the ε expansion values becomes somewhat poorer for low dimensions, $d < 3$, but even there the amplitude ratios studied here behave much better than those considered before.

The agreement found here supports the feeling that the ε expansion estimates of amplitude ratios are useful even at low dimensions. In view of this, we feel that better series and Monte Carlo estimates should be attempted for other amplitude ratios as well. We note, however, that the accuracy achieved in our present analysis, via series multiplication, is not possible for ratios like C^+/C^-, involving both high and low concentration series.

7. Conclusion

Our main results are summarised in tables 1 and 2 and in figure 1. Our new values of Δ and β at $d = 4$, 5 agree with the ε expansion and should replace older values.

Our main emphasis here is on the excellent agreement between series and ε expansion values for the various amplitude ratios. This justifies the use of ε expansion values as far as $d = 2$ ($\varepsilon = 4$) and encourages revised series and Monte Carlo studies of other ratios.

Our configuration of universality for the amplitude ratios also supports the expectation that the same ratios should be observed in more complex systems, e.g. continuum percolation, percolation of rods, cracks, etc. It would be interesting to see these checked either in computer or in real experiments. A study of several moments of the cluster size distribution (and not only of the lowest one) in such experiments should thus be encouraged.

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Appendix

Coefficients of $G_j(i, k)$ which give Γ_j via equation (5.1).

$j=3$

G(1,1) = 6.00000000000000E+00
G(2,2)= 2.40000000000000E+01
G(2,1)=-1.20000000000000E+01
G(3,3)= 8.00000000000000E+01
G(3,2)=-8.40000000000000E+01
G(3,1)= 2.20000000000000E+01
G(4,4)= 2.40000000000000E+02
G(4,3)=-3.84000000000000E+02
G(4,2)= 1.14000000000000E+02
G(4,1)= 5.40000000000000E+01
G(5,5)= 6.72000000000000E+02
G(5,4)=-1.44000000000000E+03
G(5,3)= 5.52000000000000E+02
G(5,2)= 7.72000000000000E+02
G(5,1)=-4.86000000000000E+02
G(6,6)= 1.79200000000000E+03
G(6,5)=-4.80000000000000E+03
G(6,4)= 2.48000000000000E+03
G(6,3)= 2.70000000000000E+03
G(6,2)=-1.36200000000000E+03
G(6,1)=-7.74000000000000E+02
G(7,7)= 4.60800000000000E+03
G(7,6)=-1.47840000000000E+04
G(7,5)= 1.00800000000000E+04
G(7,4)= 7.68000000000000E+03
G(7,3)= 5.55600000000000E+03
G(7,2)=-2.91060000000000E+04
G(7,1)= 1.60080000000000E+04
G(8,8)= 1.15200000000000E+04
G(8,7)=-4.30080000000000E+04
G(8,6)= 3.73440000000000E+04
G(8,5)= 1.82400000000000E+04
G(8,4)= 1.74480000000000E+04
G(8,3)=-6.09240000000000E+04
G(8,2)=-5.44740000000000E+04
G(8,1)= 7.39020000000000E+04
G(9,9)= 2.81600000000000E+04
G(9,8)=-1.19808000000000E+05
G(9,7)= 1.28128000000000E+05
G(9,6)= 3.37280000000000E+04
G(9,5)= 2.40480000000000E+04
G(9,4)= 2.10568000000000E+05
G(9,3)=-1.72962800000000E+06
G(9,2)= 2.47806600000000E+06
G(9,1)=-1.05320800000000E+06
G(10,10)= 6.75840000000000E+04
G(10,9)=-3.22560000000000E+05
G(10,8)= 4.13184000000000E+05
G(10,7)= 2.82240000000000E+04
G(10,6)=-2.64960000000000E+04
G(10,5)= 1.01521600000000E+06
G(10,4)=-4.38468000000000E+06
G(10,3)=-4.46740000000000E+05
G(10,2)= 1.16377920000000E+07
G(10,1)=-7.98146400000000E+06
G(11,11)= 1.59744000000000E+05
G(11,10)=-8.44800000000000E+05
G(11,9)= 1.26720000000000E+06
G(11,8)=-1.38240000000000E+05
G(11,7)=-3.13344000000000E+05
G(11,6)= 3.17369600000000E+06
G(11,5)= 5.89648000000000E+05
G(11,4)=-8.63136480000000E+07
G(11,3)= 2.44395668000000E+08
G(11,2)=-2.52248600000000E+08
G(11,1)= 9.02727420000000E+07

$j=4$

G(1,1)= 1.40000000000000E+01
G(2,2)= 1.00000000000000E+02
G(2,1)=-5.00000000000000E+01
G(3,3)= 5.20000000000000E+02
G(3,2)=-5.60000000000000E+02
G(3,1)= 1.50000000000000E+02
G(4,4)= 2.24000000000000E+03
G(4,3)=-3.72000000000000E+03
G(4,2)= 1.43800000000000E+03
G(4,1)= 2.36000000000000E+02
G(5,5)= 8.51200000000000E+03
G(5,4)=-1.90400000000000E+04
G(5,3)= 9.84000000000000E+03
G(5,2)= 5.62000000000000E+03
G(5,1)=-4.63000000000000E+03
G(6,6)= 2.95680000000000E+04
G(6,5)=-8.28800000000000E+04
G(6,4)= 5.58000000000000E+04
G(6,3)= 3.13920000000000E+04
G(6,2)=-3.33020000000000E+04
G(6,1)=-1.44000000000000E+02
G(7,7)= 9.60000000000000E+04
G(7,6)=-3.22560000000000E+05
G(7,5)= 2.74720000000000E+05
G(7,4)= 1.14080000000000E+05
G(7,3)=-5.79320000000000E+04
G(7,2)=-2.98210000000000E+05
G(7,1)= 1.94492000000000E+05
G(8,8)= 2.95680000000000E+05
G(8,7)=-1.15584000000000E+06
G(8,6)= 1.20960000000000E+06
G(8,5)= 2.88000000000000E+05
G(8,4)=-1.10588000000000E+05
G(8,3)=-1.44802000000000E+06
G(8,2)= 3.48172000000000E+05
G(8,1)= 5.73766000000000E+05
G(9,9)= 8.72960000000000E+05
G(9,8)=-3.88608000000000E+06
G(9,7)= 4.87065600000000E+06
G(9,6)= 2.97472000000000E+05
G(9,5)=-5.96472000000000E+05
G(9,4)=-2.75357333333328E+05
G(9,3)=-2.18909453333332E+07
G(9,2)= 3.76406393333332E+07
G(9,1)=-1.70318986666666E+07
G(10,10)= 2.48934400000000E+06
G(10,9)=-1.24185600000000E+07
G(10,8)= 1.82438400000000E+07
G(10,7)=-2.00704000000000E+06
G(10,6)=-3.59264000000000E+06
G(10,5)= 1.42067386666667E+07
G(10,4)=-1.14722997333333E+08
G(10,3)= 1.16591241333332E+08
G(10,2)= 7.59790593333339E+07
G(10,1)=-9.47677840000000E+07
G(11,11)= 6.89561600000000E+06
G(11,10)=-3.80723200000000E+07
G(11,9)= 6.43968000000000E+07
G(11,8)=-1.77500160000000E+07
G(11,7)=-1.69424640000000E+07
G(11,6)= 8.09761600000000E+07
G(11,5)=-2.41571586666664E+08
G(11,4)=-1.08476537333332E+09
G(11,3)= 4.33040877466665E+09
G(11,2)=-4.91821065466665E+09
G(11,1)= 1.83463651800000E+09

$j=5$

G(1,1)= 3.00000000000000E+01
G(2,2)= 3.60000000000000E+02
G(2,1)=-1.80000000000000E+02
G(3,3)= 2.80000000000000E+03
G(3,2)=-3.06000000000000E+03
G(3,1)= 8.30000000000000E+02
G(4,4)= 1.68000000000000E+04
G(4,3)=-2.85600000000000E+04
G(4,2)= 1.23900000000000E+04
G(4,1)= 6.90000000000000E+02
G(5,5)= 8.46720000000000E+04
G(5,4)=-1.94880000000000E+05
G(5,3)= 1.16160000000000E+05
G(5,2)= 3.35400000000000E+04
G(5,1)=-3.69420000000000E+04
G(6,6)= 3.76320000000000E+05
G(6,5)=-1.08864000000000E+06
G(6,4)= 8.42960000000000E+05
G(6,3)= 2.54140000000000E+05
G(6,2)=-4.40570000000000E+05
G(6,1)= 6.01700000000000E+04
G(7,7)= 1.52064000000000E+06
G(7,6)=-5.28192000000000E+06
G(7,5)= 5.11392000000000E+06
G(7,4)= 1.03464000000000E+06
G(7,3)=-2.02194000000000E+06
G(7,2)=-2.32245000000000E+06
G(7,1)= 1.96404000000000E+06
G(8,8)= 5.70240000000000E+06
G(8,7)=-2.30630400000000E+07
G(8,6)= 2.70883200000000E+07
G(8,5)= 1.64640000000000E+06
G(8,4)=-8.15808000000000E+06
G(8,3)=-1.96371000000000E+07
G(8,2)= 1.38728100000000E+07
G(8,1)= 2.55861000000000E+06
G(9,9)= 2.01344000000000E+07
G(9,8)=-9.27590400000000E+07
G(9,7)= 1.28970240000000E+08
G(9,6)=-1.22080000000000E+07
G(9,5)=-3.50184800000000E+07
G(9,4)=-3.87197200000000E+07
G(9,3)=-2.12803740000000E+08
G(9,2)= 4.73304890000000E+08
G(9,1)=-2.30885880000000E+08
G(10,10)= 6.76515840000000E+07
G(10,9)=-3.49240320000000E+08
G(10,8)= 5.63466240000000E+08
G(10,7)=-1.42961280000000E+08
G(10,6)=-1.53029184000000E+08
G(10,5)= 1.36533680000000E+08
G(10,4)=-1.96968552000000E+09
G(10,3)= 3.11141746000000E+09
G(10,2)=-3.80868600000000E+08
G(10,1)=-8.83263960000000E+08
G(11,11)= 2.18050560000000E+08
G(11,10)=-1.24540416000000E+09
G(11,9)= 2.29388544000000E+09
G(11,8)=-9.47566080000000E+08
G(11,7)=-6.21384960000000E+08
G(11,6)= 1.59530272000000E+09
G(11,5)=-7.78147584000000E+09
G(11,4)=-6.96158516000000E+09
G(11,3)= 6.18066781800000E+10
G(11,2)=-7.95436176800000E+10
G(11,1)= 3.11871437100000E+10

$j=6$	$j=7$	$j=8$
G(1,1)= 6.20000000000000E+01	G(1,1)= 1.26000000000000E+02	G(1,1)= 2.54000000000000E+02
G(2,2)= 1.20400000000000E+03	G(2,2)= 3.86400000000000E+03	G(2,2)= 1.21000000000000E+04
G(2,1)=-6.02000000000000E+02	G(2,1)=-1.93200000000000E+03	G(2,1)=-6.05000000000000E+03
G(3,3)= 1.36080000000000E+04	G(3,3)= 6.21600000000000E+04	G(3,3)= 2.72840000000000E+05
G(3,2)=-1.50080000000000E+04	G(3,2)=-6.89640000000000E+04	G(3,2)=-3.03920000000000E+05
G(3,1)= 4.10200000000000E+03	G(3,1)= 1.89420000000000E+04	G(3,1)= 8.37500000000000E+04
G(4,4)= 1.11216000000000E+05	G(4,4)= 6.80400000000000E+05	G(4,4)= 3.94768000000000E+06
G(4,3)=-1.92024000000000E+05	G(4,3)=-1.18742400000000E+06	G(4,3)=-6.94212000000000E+06
G(4,2)= 8.67220000000000E+04	G(4,2)= 5.70234000000000E+05	G(4,2)= 3.41955800000000E+06
G(4,1)= 2.48000000000000E+02	G(4,1)=-1.56660000000000E+04	G(4,1)=-1.59004000000000E+05
G(5,5)= 7.30464000000000E+05	G(5,5)= 5.74358400000000E+06	G(5,5)= 4.23568640000000E+07
G(5,4)=-1.71494400000000E+06	G(5,4)=-1.36785600000000E+07	G(5,4)=-1.01944480000000E+08
G(5,3)= 1.11008800000000E+06	G(5,3)= 9.33391200000000E+06	G(5,3)= 7.21121600000000E+07
G(5,2)= 1.61420000000000E+05	G(5,2)= 5.68092000000000E+05	G(5,2)= 4.54500000000000E+05
G(5,1)=-2.67526000000000E+05	G(5,1)=-1.82695800000000E+06	G(5,1)=-1.20151420000000E+07
G(6,6)= 4.09516800000000E+06	G(6,6)= 4.01533440000000E+07	G(6,6)= 3.65787136000000E+08
G(6,5)=-1.21195200000000E+07	G(6,5)=-1.20859200000000E+08	G(6,5)=-1.11523456000000E+09
G(6,4)= 1.02253200000000E+07	G(6,4)= 1.07982000000000E+08	G(6,4)= 1.03747900000000E+09
G(6,3)= 1.50975200000000E+06	G(6,3)= 5.17622000000000E+06	G(6,3)=-2.15912480000000E+07
G(6,2)=-4.61910200000000E+06	G(6,2)=-4.27449540000000E+07	G(6,2)=-3.66299830000000E+08
G(6,1)= 9.48344000000000E+05	G(6,1)= 1.06343860000000E+07	G(6,1)= 1.02655696000000E+08
G(7,7)= 2.03554560000000E+07	G(7,7)= 2.43150336000000E+08	G(7,7)= 2.67677696000000E+09
G(7,6)=-7.24711680000000E+07	G(7,6)=-8.82116928000000E+08	G(7,6)=-9.85395840000000E+09
G(7,5)= 7.62350400000000E+07	G(7,5)= 9.82870560000000E+08	G(7,5)= 1.14478830400000E+10
G(7,4)= 4.63702400000000E+06	G(7,4)=-3.77966400000000E+07	G(7,4)=-1.27006768000000E+09
G(7,3)=-3.29945800000000E+07	G(7,3)=-4.18704972000000E+08	G(7,3)=-4.64851385200000E+09
G(7,2)=-1.34576740000000E+07	G(7,2)=-3.59486820000000E+07	G(7,2)= 4.67437790000000E+08
G(7,1)= 1.77694040000000E+07	G(7,1)= 1.49276328000000E+08	G(7,1)= 1.18736937200000E+09
G(8,8)= 9.20325120000000E+07	G(8,8)= 1.31459328000000E+09	G(8,8)= 1.71848934400000E+10
G(8,7)=-3.81988992000000E+08	G(8,7)=-5.56741785600000E+09	G(8,7)=-7.39503564800000E+10
G(8,6)= 4.85102688000000E+08	G(8,6)= 7.47714844800000E+09	G(8,6)= 1.03545935680000E+11
G(8,5)=-3.28513920000000E+07	G(8,5)=-1.22236128000000E+09	G(8,5)=-2.45014112000000E+10
G(8,4)=-1.82100756000000E+08	G(8,4)=-2.94630655200000E+09	G(8,4)=-4.03172288280000E+10
G(8,3)=-1.98788188000000E+08	G(8,3)=-1.54956639600000E+09	G(8,3)=-7.73638842000000E+09
G(8,2)= 2.29119392000000E+08	G(8,2)= 2.93419186200000E+09	G(8,2)= 3.29471350920000E+10
G(8,1)=-1.04004620000000E+07	G(8,1)=-4.38869298000000E+08	G(8,1)=-7.15732219400000E+09
G(9,9)= 3.85409024000000E+08	G(9,9)= 6.48327680000000E+09	G(9,9)= 9.92025348800000E+10
G(9,8)=-1.82351769600000E+09	G(9,8)=-3.13288335360000E+10	G(9,8)=-4.87576094720000E+11
G(9,7)= 2.72708620800000E+09	G(9,7)= 4.94373116160000E+10	G(9,7)= 8.01542922752000E+11
G(9,6)=-5.84308032000000E+08	G(9,6)=-1.51936928640000E+10	G(9,6)=-3.05293129856000E+11
G(9,5)=-9.05399544000000E+08	G(9,5)=-1.72154596320000E+10	G(9,5)=-2.74919402392000E+11
G(9,4)=-6.71499181333335E+08	G(9,4)=-7.01906290400000E+09	G(9,4)=-3.57477788773330E+10
G(9,3)=-1.61411979333333E+09	G(9,3)=-7.96062615600000E+09	G(9,3)= 1.64945150386660E+10
G(9,2)= 5.31251179133331E+09	G(9,2)= 5.51798982980000E+10	G(9,2)= 5.40909953487331E+11
G(9,1)=-2.82596351466665E+09	G(9,1)=-3.23802804080000E+10	G(9,1)=-3.54582856538665E+11
G(10,10)= 1.51498547200000E+09	G(10,10)= 2.96313937920000E+10	G(10,10)= 5.24350441472000E+11
G(10,9)=-8.03516313600000E+09	G(10,9)=-1.60611010560000E+11	G(10,9)=-2.89294207488000E+12
G(10,8)= 1.38753457920000E+10	G(10,8)= 2.91936572928000E+11	G(10,8)= 5.47206792704000E+12
G(10,7)=-5.11015680000000E+09	G(10,7)=-1.34350638912000E+11	G(10,7)=-2.92590525568000E+12
G(10,6)=-4.12936944000000E+09	G(10,6)=-8.61276245760000E+10	G(10,6)=-1.51796858796800E+12
G(10,5)= 1.34580368266668E+09	G(10,5)= 2.49638454240000E+10	G(10,5)= 6.41256051986671E+11
G(10,4)=-2.74955000773332E+10	G(10,4)=-3.38815546104000E+11	G(10,4)=-3.85270330467734E+12
G(10,3)= 5.62649991533330E+10	G(10,3)= 8.54085348396000E+11	G(10,3)= 1.17096310846012E+13
G(10,2)=-2.22899845806664E+10	G(10,2)=-4.72741026576000E+11	G(10,2)=-7.81321423414062E+12
G(10,1)=-5.94065706400000E+09	G(10,1)=-7.96704367200000E+09	G(10,1)= 6.55485357848000E+11
G(11,11)= 5.64657766400000E+09	G(11,11)= 1.27021719552000E+11	G(11,11)= 2.57296151756800E+12
G(11,10)=-3.31390259200000E+10	G(11,10)=-7.62156595200000E+11	G(11,10)=-1.57227962777600E+13
G(11,9)= 6.50355763200000E+10	G(11,9)= 1.57063915008000E+12	G(11,9)= 3.36767205632000E+13
G(11,8)=-3.41675089920000E+10	G(11,8)=-9.69745835520000E+11	G(11,8)=-2.33534411601920E+13
G(11,7)=-1.67106965760000E+10	G(11,7)=-3.57979862016000E+11	G(11,7)=-6.42878902528000E+12
G(11,6)= 2.98889424960000E+10	G(11,6)= 5.79764645376000E+11	G(11,6)= 1.15331589023040E+13
G(11,5)=-1.65677665266665E+11	G(11,5)=-2.92165911755200E+12	G(11,5)=-4.63535226173465E+13
G(11,4)= 6.47774998666640E+10	G(11,4)= 3.27878897427200E+12	G(11,4)= 7.52933623148020E+13
G(11,3)= 7.56758598502667E+11	G(11,3)= 8.02962639581200E+12	G(11,3)= 7.06801348611600E+13
G(11,2)=-1.14980247803866E+12	G(11,2)=-1.53944256729200E+13	G(11,2)=-1.94709107327279E+14
G(11,1)= 4.77390622806000E+11	G(11,1)= 6.82013305558200E+12	G(11,1)= 9.28114195127580E+13

Note added in proof. We have been informed that the published version of Grassberger (1986) will contain the values $p_c = 0.160\,13 \pm 0.000\,12$ and $\beta = 0.65 \pm 0.04$ for $d = 4$, which are in substantial agreement with our estimates.

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